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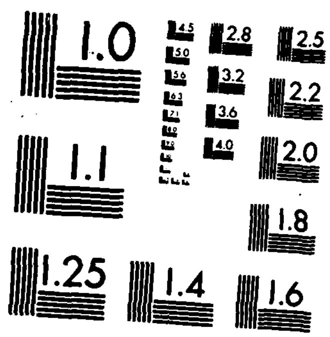

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[illegible] R. R[illegible] and Howard L. W[illegible]

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Laurence R. Riddle and Howard L. Weinert

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Recursive Linear Smoothing for Dissipative Hyperbolic Systems *

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ABSTRACT

This paper presents an efficient method of smoothing steady-state, dissipative hyperbolic systems with one spatial dimension. The observations are from point sensors placed on the system. We show that under realistic stability conditions there exists a family of finite-dimensional acausal linear systems that characterize the frequency domain behavior of the hyperbolic system. Using this characterization, we develop a smoothing algorithm that is recursive with respect to the sensors, resulting in a significant decrease in computational complexity relative to other methods. We illustrate the algorithm's performance by studying the smoothing problem for sound waves in an air-filled pipe.

1. Introduction

The purpose of this paper is to derive an algorithm for the linear least-squares smoothed estimate of inputs or state variables in a dissipative hyperbolic system described by a vector first-order partial differential equation with boundary and initial conditions [9] . Examples of such systems are those described by the telegrapher's equation, the vibrating Timoshenko beam equation, and the wave equation. Our smoothing algorithm can be used, for exam-

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ple, to estimate the radiated sound from a vibrating object given observations at discrete points in space [3]. The systems we consider here have one spatial dimension and are operating in temporal steady-state. The observations are taken by N_s point sensors distributed non-uniformly across the system. We assume either (a)that the observation interval is long enough to reliably compute the Fourier transform with respect to time, and to cause different frequency components to become uncorrelated with each other, or (b) that the relevant random processes are periodic in time and are observed over an interval that is a multiple of this period.

To solve this smoothing problem we first Fourier transform the observations with respect to time. We then have a set of uncoupled spatial smoothing problems over space, indexed by the frequency variable ,in which the underlying *models are finite-dimensional well-posed acausal linear systems* [4]-[5]. The acausal linear system smoothing problems are solved by the method of complementary models [1],[12] after which one may use an inverse Fourier transform to recover the estimates as functions of space and time. The resulting algorithm is recursive with respect to the sensors, and thus offers a significant decrease in complexity relative to other methods.

2. Dissipative Hyperbolic Systems

In many signal processing problems, one has measurements of the output of a system described by a wave-like (hyperbolic) partial differential equation. Physically, a dissipative hyperbolic (DH) system [9] is a model for a wave bearing structure that has internal energy loss due to distributed or boundary damping. Examples of such systems are vibrating strings, beams, transmission lines, acoustical and electromagnetic waveguides, etc. We will consider DH systems with one spatial variable.

A DH system is described by a vector first order-system of partial differential equations

$$\frac{\partial}{\partial t}m(x,t) = \Lambda(x)\frac{\partial}{\partial x}m(x,t)+G(x)m(x,t)+\epsilon(x,t) \ , \ x \in [0,L] \ , \ t \geq t_0 \qquad (2.1)$$

with boundary conditions

$$H_0 m(0,t) = d_1(t) \ , \ H_L m(L,t) = d_2(t) \qquad (2.2)$$

and initial condition

$$m(x,t_0) = m_0(x) \qquad (2.3)$$

where $m(x,t)$ is the $n \times 1$ state vector, $\epsilon(x,t)$ is the input field, $\Lambda(x)$ is a symmetric, continuously differentiable matrix with constant rank r, $G(x)$ is a continuous matrix, $d_1(t)$ and $d_2(t)$ are $n/2 \times 1$ boundary inputs, and H_0 and H_L are matrices of bounded, linear, causal, shift-invariant operators. All quantities in Eqs. (2.1)-(2.3) are real. Note that according to the Bochner-Chandrasekharan theorem [2], the boundary operators H_0 and H_L are such that their operation in the frequency domain is multiplicative; i.e., the following transform relations hold:

$$H_0 m(0,t) \Leftrightarrow H_0(j\omega) m(0,j\omega)$$
$$H_L m(L,t) \Leftrightarrow H_L(j\omega) m(L,j\omega)$$

where $H_0(j\omega)$ and $H_L(jw)$ are complex valued $n/2 \times n$ matrices.

A DH system will satisfy [9]

$$G(x)+G'(x) - \frac{\partial}{\partial x}\Lambda(x) \leq 0 \ \text{ for all } x \in [0,L] \qquad (2.4)$$

$$m'(L,t)\Lambda(L)m(L,t) - m'(0,t)\Lambda(0)m(0,t) \leq 0 \text{ , for all} t \geq t_0 \quad (2.5)$$

when $d_1 = d_2 = 0$. These conditions ensure that when $\epsilon = d_1 = d_2 = 0$

$$\frac{\partial}{\partial t}\|m(x,t)\|^2 \leq 0 \text{ , for all} t \geq t_0 \quad (2.6)$$

where

$$\|m(x,t)\|^2 = \int_0^L m'(x,t)m(x,t)\,dx$$

To see this, pre-multiply Eq (2.1) (with $\epsilon = 0$) by $m'(x,t)$ and add the transpose of the result, to obtain

$$\frac{\partial}{\partial t}m'(x,t)m(x,t) = \frac{\partial}{\partial x}(m'(x,t)\Lambda(x)m(x,t)) + m'(x,t)[G(x)+G'(x) - \frac{\partial}{\partial x}\Lambda(x)]m(x,t)$$

It then follows that

$$\frac{\partial}{\partial t}\|m(x,t)\|^2 = \int_0^L m'(x,t)(G(x)+G'(x) - \frac{\partial}{\partial x}\Lambda(x))m(x,t)\,dx + m'(L,t)\Lambda(L)m(L,t) - m'(0,t)\Lambda(0)m(0,t) \leq 0$$

An example of a DH system is the damped wave equation

$$u_{tt} - c^2 u_{xx} + \gamma u_t = \epsilon$$

with boundary conditions (damped supports)

$$u_t(0,t) - k_0 u_x(0,t) = d_1(t)$$

$$u_t(L,t) + k_L u_x(L,t) = d_2(t), \quad k_0, k_L \geq 0$$

and initial conditions

$$u_x(x,0) = g_1(x) \quad , u_t(x,0) = g_2(x)$$

Setting

$$m_1(x,t) = cu_x(x,t) \ , \ m_2(x,t) = u_t(x,t)$$

one obtains

$$\frac{\partial}{\partial t}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\epsilon$$

$$\begin{bmatrix} -k_0/c & \vdots & 1 \end{bmatrix}\begin{bmatrix} m_1(0,t) \\ m_2(0,t) \end{bmatrix} = d_1(t)$$

$$\begin{bmatrix} k_L/c & \vdots & 1 \end{bmatrix}\begin{bmatrix} m_1(L,t) \\ m_2(L,t) \end{bmatrix} = d_2(t)$$

$$m_0(x) = \begin{bmatrix} cg_1(x) \\ g_2(x) \end{bmatrix}$$

Note that

$$G + G' - \frac{\partial}{\partial x}\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -2\gamma \end{bmatrix} \leq 0$$

and when $d_1 = d_2 = 0$

$$m'(L,t)\Lambda\, m(L,t) - m'(0,t)\Lambda\, m(0,t) = -2(k_0 m_1^2(0,t) + k_L m_1^2(L,t)) \leq 0.$$

We will assume that the DH system is asymptotically stable. That is, if $m(x,t)$ is the solution of Eq (2.1) with $\epsilon = d_1 = d_2 = 0$ then $\|m(x,t)\| \to 0$ as $t \to \infty$. If we had assumed that the inequality in Eq (2.4) was strict, then the inequality in Eq (2.6) would be strict also, giving us asymptotic stability. However, we want our results to apply to systems like the above example that do not have a strict inequality in Eq (2.4), but still are asymptotically stable. One should note that a system that has normal modes, i.e., non-decaying responses to initial conditions, is not asymptotically stable. In practice, however, one always has dissipative elements in the system and these elements should be retained in the model to ensure a proper formulation.

Yet another stability assumption is required for the smoothing problem when the Λ matrix is singular. In this case it is shown in Appendix A that each member of the family of finite-dimensional systems comprised of those state variables associated with the zero eigenvalues of Λ has poles with only nonpositive real parts. In order for the results of this paper to apply, we must assume that all of these poles in fact have negative real parts. Again, if the inequality in Eq (2.4) is strict, then this assumption is automatically satisfied. Physically, these state variables correspond to damping of the hyperbolic system. This stability assumption can be shown to ensure that an acausal linear system representation exists and is well-posed. The well-posedness of the acausal linear system then implies that the resulting smoother is well-posed.

3. Problem Statement and Construction of the Acausal System

We wish to determine the linear least-squares smoothed estimates of the state $m(x,t)$ and inputs $\epsilon(x,t),d_1(t),d_2(t)$ of the DH system (2.1)-(2.3) given observations

$$y_k(t) = Cm(x_k,t)+w_k(t)$$

at specific points x_k along the system, where C is $p \times n$ and

$$t \in \left[-T/2\ ,\ T/2\right],\ \ k = 1,2,\cdots,N_s$$

$$0<x_1<x_2<\cdots<x_{N_s}<L$$

We assume that $\epsilon(x,t), d_1(t), d_2(t)$ and observation noise $w_k(t)$ are zero mean and wide-sense stationary in time, that $t_0 = -\infty$ and $m_0(x) = 0$. The signals $m(x,t)$ and $y_k(t)$ are then also wide-sense stationary . We also assume that

$$E\epsilon(x,t)\epsilon'(z,s) = Q(x,t-s)\delta(x-z)$$

$$Ew_i(t)w_j'(s) = R_w(t-s)\delta_{ij}$$

$$Ev(t)v'(s) = \Pi_v(t-s) \text{ where } v(t) = [d_1'(t) \;\; d_2'(t)]'$$

$$E\epsilon(x,t)w_i'(s) = \;\; E\epsilon(x,t)v'(s) = \;\; Ew_i(t)v'(s) = 0$$

We will use a Fourier series expansion (in time) of the signals $y_k(t)$, $m(x,t)$,etc., over the interval $[-T/2 \, , \, T/2]$, and denote the Fourier coefficients by $y_k(j\omega), m(x,j\omega)$, etc., where $\omega = 2\pi l/T$; $l = 0, \pm 1, \pm 2, \cdots$. These coefficients can be computed using:

$$y_k(j\omega) = \frac{1}{T}\int_{-T/2}^{T/2} y_k(t)e^{-j\omega t}\,dt$$

Note that this integral can be evaluated at $\omega = 2\pi l/T$ using discrete-time data and an FFT if the signal is band-limited. The stationarity assumption implies that the Fourier coefficients are uncorrelated at different frequencies provided one of the following conditions holds [8] : (1) the covariances $Q(x,\tau)$, $R_w(\tau)$, $\Pi_v(\tau)$ are periodic in τ with period T ; or (2) the observation interval T is long and the covariances go to zero as $\tau \to \infty$. We note that in many vibration and acoustical problems, the signals (and hence their covariances) are periodic. In the following we will assume that one of these conditions holds, in which case the original two-dimensional (space-time) estimation problem can be replaced by a family of independent one-dimensional (spatial) estimation problems. In particular, for each fixed ω, we must estimate $m(x,j\omega), v(j\omega), \epsilon(x,jw)$ given $y_k(j\omega), k = 1,2, \cdots N_s$, and then inverse transform the results to get the time domain estimates. Moreover, as we shall now show, $m(x,j\omega)$ is the state vector of a *finite- dimensional* acausal linear system.

The model for $m(x,j\omega)$ is given by the Fourier expansion of Eqs. (2.1)-(2.3):

$$j\omega m(x,j\omega) = \Lambda(x)\frac{\partial}{\partial x}m(x,j\omega) + G(x)m(x,j\omega) + \epsilon(x,j\omega) \tag{3.1}$$

$$V_0 m(0,j\omega) + V_L m(L,j\omega) = v(jw) \tag{3.2}$$

$$y_k(j\omega) = Cm(x_k,j\omega) + w_k(j\omega) \ , \ k = 1,2, \cdots, N_s \tag{3.3}$$

where

$$V_0 = \begin{bmatrix} H_0(j\omega) \\ \cdots \\ 0 \end{bmatrix}, V_L = \begin{bmatrix} 0 \\ \cdots \\ H_L(j\omega) \end{bmatrix}$$

$$E\epsilon(x,j\omega)\epsilon^*(z,j\omega) = Q(x,j\omega)\delta(x-z)$$

$$E\ v(jw)v^*(j\omega) = \Pi_v(j\omega);\ Ew_k(j\omega)w_i^*(j\omega) = R_w(j\omega)\delta_{ki}$$

$* =$ conjugate transpose , $Q(x,j\omega), \Pi_v(j\omega)\ R_w(j\omega)$ are the Fourier coefficients of $Q(x,\tau), \Pi_v(\tau)$, $R_w(\tau)$, which when T is large can be approximated by dividing the spectral densities of ϵ ,v, and w_k by T. We will assume that $R_w(j\omega)$ is invertible.

If $\Lambda(x)$ is invertible for all x, we can write Eq (3.1) as

$$\frac{\partial}{\partial x} m(x,j\omega) = A(x,j\omega) m(x,j\omega) + B(x) \epsilon(x,j\omega) \tag{3.4}$$

where

$$A(x,j\omega) = \Lambda^{-1}(x)\{j\omega I - G(x)\}$$

$$B(x) = -\Lambda^{-1}(x)$$

Eqs. (3.4),(3.2),(3.3) are a family of acausal linear systems indexed by ω. It is shown in Appendix A that a similar acausal linear system representation can be obtained even when $\Lambda(x)$ is not invertible, provided the stability assumptions discussed earlier hold.

We must consider the well-posedness of the acausal linear system description of the DH system. An acausal linear system is well-posed if there exist no nonzero solutions to an undriven system.

Theorem: The acausal linear system of Eqs (3.2)-(3.4) is well-posed for all ω.

Proof: Suppose there exists an ω_0 such that Eqs (3.2)-(3.4) are not well-posed. There then exists $m(x,j\omega_0) \neq 0$ satisfying

$$\frac{\partial}{\partial x} m(x,j\omega_0) = A(x,j\omega_0) m(x,j\omega_0)$$

$$V_0 m(0,j\omega_0) + V_L m(L,j\omega_0) = 0$$

or, equivalently,

$$j\omega_0 m(x,j\omega_0) = \Lambda(x) \frac{\partial}{\partial x} m(x,j\omega_0) + G(x) m(x,j\omega_0)$$

$$H_0(j\omega_0)m(0,j\omega_0) = H_L(j\omega_0)m(L,j\omega_0) = 0$$

It is easily checked that if $\Psi(x,t) = e^{j\omega_0 t}m(x,j\omega_0)$, then $\Psi(x,t)$ satisfies Eqs (2.1) - (2.2) with zero inputs:

$$\frac{\partial}{\partial t}\Psi(x,t) = \Lambda(x)\frac{\partial}{\partial x}\Psi(x,t)+G(x)\Psi(x,t)$$

$$H_0\Psi(0,t) = H_L\Psi(L,t) = 0$$

Since $\|\Psi(x,t)\|$ does not go to zero as $t\rightarrow\infty$, we have a contradiction of the asymptotic stability assumption.

4. Smoothing the Acausal Linear System

In this section we show how to solve the smoothing problem for our acausal linear system. Although this paper concentrates on DH systems, many parabolic type equations can also be written in an acausal linear system form. The smoother is derived by means of complementary models, introduced by Weinert and Desai [12], and extended by Adams , Willsky and Levy [1]. The derivation differs significantly from that in [1] due to the possible singularity of Q and Π_v , and the fact that the observations are discrete. In what follows, the ω dependence in Eqs. (3.2)-(3.4) will be suppressed.

A solution to Eqs. (3.2),(3.4) is

$$m(x) = \Phi(x,0)F^{-1}v+\int_0^L G(x,z)B(z)\epsilon(z)\,dz \tag{4.1a}$$

where the state transition matrix Φ satisfies

$$\frac{\partial}{\partial x}\Phi(x,z) = A(x)\Phi(x,z), \quad \Phi(z,z) = I \tag{4.1b}$$

and the Green function is given by

$$G(x,z) = \begin{cases} \Phi(x,0)F^{-1}V_0\Phi(0,z) & \text{if } z<x \\ -\Phi(x,0)F^{-1}V_L\Phi(L,z) & \text{if } z>x \end{cases} \tag{4.1c}$$

and the matrix F satisfies

$$F = V_0+V_L\Phi(L,0) \tag{4.1d}$$

The well-posedness of the foregoing acausal linear system guarantees the invertibility of F [4]-[5].

If rank $\Pi_v = q_1$ we can write a full rank factorization of Π_v as

$$\Pi_v = MM^*$$

where M is $n\times q_1$; in other words,

$$v = M\mu \ , \ E\mu\mu^* = I_{q_1}$$

Similarly, if rank $Q(x) = q_2$, we can write

$$Q(x) = S(x)S^*(x)$$

where $S(x)$ is $n\times q_2$; thus

$$\epsilon(x) = S(x)\rho(x) \ , \ \ E\rho(x)\rho^*(z) = I_{q_2}\delta(x-z)$$

Using Eq (4.1a), we can write Eq (3.3) as

$$y_j = C\Phi(x_j,0)F^{-1}M\mu+\int_0^L CG(x_j,z)B(z)S(z)\rho(z)dz+w_j \tag{4.2}$$

Eq (4.2) relates the observations to the underlying variables $\mu,\rho(\cdot),\{w_j\}$, which together span a Hilbert space H. If Y is the Hilbert space spanned by the observations $\{y_j\}$, then it is shown in Appendix B that $y_c(\cdot)$ and θ defined below span the orthogonal complement $Y^{\perp}$:

$$y_c(x) = \rho(x) - S^*(x)B^*(x)\lambda(x) \tag{4.3a}$$

$$\theta = \mu - M^*\{F^*\}^{-1}(\lambda(0) - \Phi^*(L,0)\lambda(L)) \tag{4.3b}$$

where

$$\lambda(x) = \sum_{k=1}^{N_s} G^*(x_k,x)C^*R_w^{-1}w_k \tag{4.4}$$

Now Eq. (4.4) implies

$$\frac{d}{dx}\lambda(x) = -A^*(x)\lambda(x) \; ; \; x \neq x_j \tag{4.5a}$$

$$\lambda(x_j-) = \lambda(x_j+) + C^*R_w^{-1}w_j \, , \; j = 1,2, \cdots, N_s \tag{4.5b}$$

In order to make Eqs. (4.5) equivalent to Eq. (4.4) a boundary condition of the following form is needed:

$$K_0\lambda(0) + K_L\lambda(L) = 0 \tag{4.5c}$$

where K_0 and K_L are $n \times n$ matrices . Eqs. (4.1c) and (4.4) imply that Eq. (4.5c) holds if

$$K_0V_0^* - K_LV_L^* = 0 \, . \tag{4.6}$$

Furthermore, if $K_0\Phi^*(L,0) + K_L$ is invertible, then Eqs. (4.5) will be well-posed. If we take

$$K_0 = V_L^*\{F^*\}^{-1}$$

$$K_L = I - V_L^*\{F^*\}^{-1}\Phi^*(L,0)$$

then both the invertibility condition and Eq (4.6) will be satisfied. With this choice of K_0 and K_L, the acausal linear system (4.3), (4.5) is complementary to (3.2)-(3.4).

Solving Eq. (4.3a) for $\rho(x)$ and substituting into Eq. (3.4) gives

$$\frac{d}{dx}m(x) = A(x)m(x)+B(x)S(x)(S^*(x)B^*(x)\lambda(x)+y_c(x)) \qquad (4.7)$$

Likewise, solving Eq. (3.3) for w_j and substituting into Eq. (4.5b) gives

$$\lambda(x_j-) = \lambda(x_j+)+C^*R_w^{-1}(y_j-Cm(x_j)) \qquad (4.8)$$

Multiplying Eq. (4.3b) by M and re-arranging, we get

$$v = M\theta+\Pi_v\{F^*\}^{-1}(\lambda(0)-\Phi^*(L,0)\lambda(L)) \qquad (4.9)$$

Combining Eqs. (3.2),(4.5),(4.7)-(4.9) gives the Hamiltonian system

$$\frac{d}{dx}\begin{bmatrix} m(x) \\ \lambda(x) \end{bmatrix} = \begin{bmatrix} A & BQB^* \\ 0 & -A^* \end{bmatrix}\begin{bmatrix} m(x) \\ \lambda(x) \end{bmatrix} + \begin{bmatrix} BS \\ 0 \end{bmatrix} y_c(x)\ ,\ x \neq x_j \qquad (4.10a)$$

$$\lambda(x_j-) = \lambda(x_j+)-C^*R_w^{-1}Cm(x_j)+C^*R_w^{-1}y_j \qquad (4.10b)$$

$$\begin{bmatrix} M\theta \\ 0 \end{bmatrix} = \begin{bmatrix} V_0 & -\Pi_v\{F^*\}^{-1} \\ 0 & K_0 \end{bmatrix}\begin{bmatrix} m(0) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} V_L & \Pi_v\{F^*\}^{-1}\Phi^*(L,0) \\ 0 & K_L \end{bmatrix}\begin{bmatrix} m(L) \\ \lambda(L) \end{bmatrix} \qquad (4.10c)$$

By projecting the random quantities in the Hamiltonian onto the subspace Y spanned by the observations, we obtain the estimate Hamiltonian:

$$\frac{d}{dx}\begin{bmatrix} \hat{m}(x) \\ \hat{\lambda}(x) \end{bmatrix} = \begin{bmatrix} A & BQB^* \\ 0 & -A^* \end{bmatrix}\begin{bmatrix} \hat{m}(x) \\ \hat{\lambda}(x) \end{bmatrix},\ x \neq x_j \qquad (4.11a)$$

$$\hat{\lambda}(x_j-) = \hat{\lambda}(x_j+)-C^*R_w^{-1}C\hat{m}(x_j)+C^*R_w^{-1}y_j \qquad (4.11b)$$

$$0 = \begin{bmatrix} V_0 & -\Pi_v\{F^*\}^{-1} \\ 0 & K_0 \end{bmatrix}\begin{bmatrix} \hat{m}(0) \\ \hat{\lambda}(0) \end{bmatrix} + \begin{bmatrix} V_L & \Pi_v\{F^*\}^{-1}\Phi^*(L,0) \\ 0 & K_L \end{bmatrix}\begin{bmatrix} \hat{m}(L) \\ \hat{\lambda}(L) \end{bmatrix} \qquad (4.11c)$$

where ˆ denotes projection onto Y. To obtain $\hat{\epsilon}(x)$ and $\hat{v}$, project onto Y in Eq. (4.3a) (after multiplying by S), and in Eq. (4.9):

$$\hat{\epsilon}(x) = Q(x)B^*(x)\hat{\lambda}(x) \tag{4.12a}$$

$$\hat{v} = \Pi_v\{F^*\}^{-1}(\hat{\lambda}(0) - \Phi^*(L,0)\hat{\lambda}(L)) \tag{4.12b}$$

To prove that Eqs (4.11) are well-posed, assume that $y_j = 0$ for all j. Now $\hat{m}(x)$ is the linear-least squares estimate of $m(x)$ based on observations that are all zero, so $\hat{m}(x) = Em(x) = 0$, the last equality following from the well-posedness of Eqs. (3.2)-(3.4). Moreover, since $\hat{\lambda}(x)$ satisfies the unforced version of Eqs. (4.5), which are well-posed, $\hat{\lambda}(x) = 0$ for all x , and hence Eqs. (4.11) are well-posed.

5. Recursive Solution to the Estimate Hamiltonian

To solve the estimate Hamiltonian (4.11), we will first diagonalize the dynamics with the following change of variables :

$$\begin{bmatrix} \Psi_f(x) \\ \Psi_b(x) \end{bmatrix} = T(x)\begin{bmatrix} \hat{m}(x) \\ \hat{\lambda}(x) \end{bmatrix}$$

where

$$T(x) = \begin{bmatrix} \theta_f(x) & -I \\ \theta_b(x) & I \end{bmatrix}$$

$$T^{-1}(x) = \begin{bmatrix} I & I \\ -\theta_b(x) & \theta_f(x) \end{bmatrix}\begin{bmatrix} P(x) & 0 \\ 0 & P(x) \end{bmatrix}$$

$$P(x) = (\theta_f(x) + \theta_b(x))^{-1}$$

and $\theta_f(x)$ and $\theta_b(x)$ satisfy

$$-\frac{d}{dx}\theta_f = \theta_f A + A^*\theta_f + \theta_f BQB^*\theta_f \ , x \neq x_j$$

$$-\frac{d}{dx}\theta_b = \theta_b A + A^*\theta_b - \theta_b BQB^*\theta_b \ , x \neq x_j$$

with any positive-definite boundary values $\theta_f(0)$ and $\theta_b(L)$. The new variables Ψ_f and Ψ_b satisfy

$$\frac{d}{dx}\begin{bmatrix}\Psi_f(x)\\ \Psi_b(x)\end{bmatrix} = \begin{bmatrix}A_f & 0\\ 0 & A_b\end{bmatrix}\begin{bmatrix}\Psi_f(x)\\ \Psi_b(x)\end{bmatrix}, \ x \neq x_j \tag{5.1a}$$

$$[V_f^0 \ \ V_b^0]\begin{bmatrix}\Psi_f(0)\\ \Psi_b(0)\end{bmatrix} + [V_f^L \ \ V_b^L]\begin{bmatrix}\Psi_f(L)\\ \Psi_b(L)\end{bmatrix} = 0 \tag{5.1b}$$

where

$$A_f(x) = -(A^*(x) + \theta_f(x)B(x)Q(x)B^*(x))$$

$$A_b(x) = -(A^*(x) - \theta_b(x)B(x)Q(x)B^*(x))$$

$$[V_f^0 \ \ V_b^0] = \begin{bmatrix} V_0 + \Pi_v\{F^*\}^{-1}\theta_b(0) & V_0 - \Pi_v\{F^*\}^{-1}\theta_f(0) \\ -K_0\theta_b(0) & K_0\theta_f(0)\end{bmatrix}\begin{bmatrix}P(0) & 0\\ 0 & P(0)\end{bmatrix}$$

$$[V_f^L \ \ V_b^L] = \begin{bmatrix} V_L - \Pi_v\{F^*\}^{-1}\Phi^*(L,0)\theta_b(L) & V_L + \Pi_v\{F^*\}^{-1}\Phi^*(L,0)\theta_f(L) \\ -K_L\theta_b(L) & K_L\theta_f(L)\end{bmatrix}\begin{bmatrix}P(L) & 0\\ 0 & P(L)\end{bmatrix}$$

We must now specify the evolution of $\Psi_f, \Psi_b, \theta_f, \theta_b$ at $x = x_j \ , j = 1, \cdots, N_s$. If we choose

$$\theta_f(x_j+) = \theta_f(x_j-) + C^*R_w{}^{-1}C$$

$$\theta_b(x_j-) = \theta_b(x_j+) + C^*R_w{}^{-1}C$$

then one can show that

$$\Psi_f(x_j+) = \Psi_f(x_j-) + C^* R_w^{-1} y_j \tag{5.2a}$$

$$\Psi_b(x_j-) = \Psi_b(x_j+) + C^* R_w^{-1} y_j \tag{5.2b}$$

Eqs (5.1)-(5.2) are in a form that can be solved recursively. In terms of $\Psi_f(0)$ and $\Psi_b(L)$, a solution to Eqs. (5.1)-(5.2) is

$$\Psi_f(x) = \Phi_{A_f}(x,0)\Psi_f(0) + \Psi_f^0(x) \tag{5.3a}$$

$$\Psi_b(x) = \Phi_{A_b}(x,L)\Psi_b(L) + \Psi_b^0(x) \tag{5.3b}$$

where

$$\frac{d}{dx}\Phi_{A_f}(x,0) = A_f(x)\Phi_{A_f}(x,0) \ ; \ \Phi_{A_f}(0,0) = I$$

$$\frac{d}{dx}\Phi_{A_b}(x,L) = A_b(x)\Phi_{A_b}(x,L) \ ; \ \Phi_{A_b}(L,L) = I$$

$$\frac{d}{dx}\Psi_f^0(x) = A_f(x)\Psi_f^0(x) \ , \ x \neq x_j \tag{5.4a}$$

$$\frac{d}{dx}\Psi_b^0(x) = A_b(x)\Psi_b^0(x) \ , \ x \neq x_j \tag{5.4b}$$

$$\Psi_f^0(0) = \Psi_b^0(L) = 0 \tag{5.4c}$$

$$\Psi_f^0(x_j+) = \Psi_f^0(x_j-) + C^* R_w^{-1} y_j \ , \ j = 1, \cdots, N_s \tag{5.4d}$$

$$\Psi_b^0(x_j-) = \Psi_b^0(x_j+) + C^* R_w^{-1} y_j \ , \ j = 1, \cdots, N_s \tag{5.4e}$$

Setting $x = L$ in Eq. (5.3a) and $x = 0$ in Eq. (5.3b), and using Eq. (5.1b) we obtain

$$\begin{bmatrix} \Psi_f(0) \\ \Psi_b(L) \end{bmatrix} = -F_{fb}^{-1}\{V_f^L \Psi_f^0(L) + V_b^0 \Psi_b^0(0)\}$$

where

$$F_{fb} = \left[V_f^0 + V_f^L \Phi_{A_f}(L,0) \ \vdots \ V_b^L + V_b^0 \Phi_{A_b}(0,L) \right]$$

A recursive solution to Eqs (5.1)-(5.2) is therefore given by:

$$\begin{bmatrix} \Psi_f(x) \\ \Psi_b(x) \end{bmatrix} = - \begin{bmatrix} \Phi_{A_f}(x,0) & 0 \\ 0 & \Phi_{A_b}(x,L) \end{bmatrix} F_{fb}^{-1} \{ V_f^L \Psi_f^0(L) + V_b^0 \Psi_b^0(0) \}$$

$$+ \begin{bmatrix} \Psi_f^0(x) \\ \Psi_b^0(x) \end{bmatrix} \qquad (5.5)$$

in conjunction with Eqs. (5.4).

We now show that F_{fb} is invertible. When $y_j = 0$, $j = 1, \cdots, N_s$, any solution to Eqs (5.1)-(5.2) will satisfy

$$\Psi_f(x) = \Phi_{A_f}(x,0)\beta_1$$

$$\Psi_b(x) = \Phi_{A_b}(x,L)\beta_2$$

where

$$F_{fb} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$$

If F_{fb} were not invertible, we could generate non-zero solutions to Eqs. (5.1)-(5.2) and hence to Eqs (4.11) (via $T^{-1}(x)$) with $y_j = 0$, contradicting the well-posedness of Eqs (4.11).

6. Smoothing Error Covariance

We wish to calculate the time domain error covariance

$$P_m(x,t) = E[\tilde{m}(x,t)\tilde{m}'(x,t)]$$

where $\tilde{m} = m - \hat{m}$. Using Fourier transforms, one can show that

$$P_m(x,t) = P_m(x,0) = \sum_{l=-\infty}^{\infty} P_m(x,j2\pi l/T)$$

where

$$P_m(x,j\omega) = E[\tilde{m}(x,j\omega)\tilde{m}^*(x,j\omega)] \ .$$

A dynamical equation governing $\tilde{m}(x,j\omega)$ is obtained by projecting the Hamiltonian system (4.10) onto $Y^{\perp}$, and using the definitions of y_c and θ:

$$\frac{d}{dx}\begin{bmatrix}\tilde{m}(x)\\ -\hat{\lambda}(x)\end{bmatrix} = \begin{bmatrix}A & BQB^*\\ 0 & -A^*\end{bmatrix}\begin{bmatrix}\tilde{m}(x)\\ -\hat{\lambda}(x)\end{bmatrix} + \begin{bmatrix}B\epsilon\\ 0\end{bmatrix}, \ x \neq x_j \tag{6.1a}$$

$$-\hat{\lambda}(x_j-) = -\hat{\lambda}(x_j+) - C^*R_w^{-1}C\tilde{m}(x_j) - C^*R_w^{-1}w_j \tag{6.1b}$$

$$\begin{bmatrix}v\\ 0\end{bmatrix} = \begin{bmatrix}V_0 & -\Pi_v\{F^*\}^{-1}\\ 0 & K_0\end{bmatrix}\begin{bmatrix}\tilde{m}(0)\\ -\hat{\lambda}(0)\end{bmatrix} + \begin{bmatrix}V_L & \Pi_v\{F^*\}^{-1}\Phi^*(L,0)\\ 0 & K_L\end{bmatrix}\begin{bmatrix}\tilde{m}(L)\\ -\hat{\lambda}(L)\end{bmatrix} \tag{6.1c}$$

Here again the ω dependence has been suppressed. Note that these equations are similar to Eqs. (4.11) and can therefore be solved using the same diagonalizing change of variables. Thus, if

$$\begin{bmatrix}e_f(x)\\ e_b(x)\end{bmatrix} = T(x)\begin{bmatrix}\tilde{m}(x)\\ -\hat{\lambda}(x)\end{bmatrix}$$

then

$$\begin{bmatrix}e_f(x)\\ e_b(x)\end{bmatrix} = \Phi_{fb}(x)F_{fb}^{-1}[v - V_f^L e_f^0(L) - V_b^0 e_b^0(0)] + \begin{bmatrix}e_f^0(x)\\ e_b^0(x)\end{bmatrix}$$

where

$$\frac{d}{dx}e_f^0(x) = A_f(x)e_f^0(x) + \theta_f(x)B(x)\epsilon(x) \ , \ x \neq x_j$$

$$\frac{d}{dx}e_b^0(x) = A_b(x)e_b^0(x) + \theta_b(x)B(x)\epsilon(x) \ , \ x \neq x_j$$

$$e_f^0(0) = e_b^0(L) = 0$$

$$e_f^0(x_j+) = e_f^0(x_j-) - C^* R_w^{-1} w_j \ , \ j = 1, \cdots, N_s$$

$$e_b^0(x_j-) = e_b^0(x_j+) - C^* R_w^{-1} w_j \ , \ j = 1, \cdots, N_s$$

Letting

$$\Phi_{fb}(x) = \begin{bmatrix} \Phi_{A_f}(x,0) & 0 \\ 0 & \Phi_{A_b}(x,L) \end{bmatrix}$$

then

$$\Theta(x) = E\begin{bmatrix} e_f(x) \\ e_b(x) \end{bmatrix}[e_f^*(x) e_b^*(x)] = \begin{bmatrix} \Theta_{11}(x) & \Theta_{12}(x) \\ \Theta_{21}(x) & \Theta_{22}(x) \end{bmatrix}$$

$$P_m(x,j\omega) = P(x)[\Theta_{11}(x)+\Theta_{22}(x)+\Theta_{12}(x)+\Theta_{21}(x)]P(x)$$

With the following definitions:

$$\Pi_f(x) = E[e_f^0(x) e_f^{0*}(x)]$$

$$\Pi_b(x) = E[e_b^0(x) e_b^{0*}(x)]$$

$$R_{fb}(x,y) = E[e_f^0(x) e_b^{0*}(y)]$$

$\Theta(x)$ can be written as

$$\Theta(x) = \Phi_{fb}(x) F_{fb}^{-1}\Big[\Pi_v + V_f^L \Pi_f(L) V_f^{L*} + V_f^L R_{fb}(L,0) V_b^{0*} + V_b^0 \Pi_b(0) V_b^{0*} + V_b^0 R_{fb}^*(L,0) V_f^{L*}\Big](F_{fb}^*)^{-1}\Phi_{fb}^*(x) - \Phi_{fb}(x) F_{fb}^{-1} J(x) - J^*(x)(F_{fb}^*)^{-1}\Phi_{fb}^*(x) + \begin{bmatrix} \Pi_f(x) & 0 \\ 0 & \Pi_b(x) \end{bmatrix}$$

where

$$J(x) = V_f^L\left[\Phi_{A_f}(L,x)\Pi_f(x) \vdots R_{fb}(L,x)\right] + V_b^0\left[R_{fb}^*(x,0) \vdots \Pi_b(x)\Phi_{A_b}^*(0,x)\right]$$

By direct evaluation, one can obtain the following formulas for R_{fb} , Π_f , Π_b:

$$R_{fb}(x,y) = -\int_y^x \Phi_{A_f}(x,\sigma)\theta_f(\sigma)B(\sigma)Q(\sigma)B^*(\sigma)\theta_b(\sigma)\Phi_{A_b}^*(y,\sigma)\,d\sigma,\ x \geq y$$

$$\frac{d}{dx}\Pi_f(x) = A_f\Pi_f(x) + \Pi_f(x)A_f^* + \theta_f BQB^*\theta_f,\ \ x \neq x_j$$

$$\frac{d}{dx}\Pi_b(x) = A_b\Pi_b(x) + \Pi_b(x)A_b^* - \theta_b BQB^*\theta_b\ ,\ x \neq x_j$$

$$\Pi_f(0) = \Pi_b(L_1) = 0$$

$$\Pi_f(x_j+) = \Pi_f(x_j-) + C^*R_w^{-1}C\ ,\ j = 1, \cdots, N_s$$

$$\Pi_b(x_j-) = \Pi_b(x_j+) + C^*R_w^{-1}C\ ,\ j = 1, \cdots, N_s$$

7. Separable Boundary Conditions

In this section, we show that if the boundary inputs $d_1(t)$ and $d_2(t)$ of the DH system are uncorrelated, and if the resulting block diagonal Π_v is invertible, the smoother and error covariance formulas simplify considerably. Under these conditions, $V_0^*\Pi_v^{-1}V_L = 0$, in which case the acausal linear system has separable boundary conditions. To examine the filter and smoothing error covariance for separable boundary conditions, premultiply Eq. (4.11c) by

$$\begin{bmatrix} V_0^*\Pi_v^{-1} & -\Phi^*(L,0) \\ V_L^*\Pi_v^{-1} & I \end{bmatrix}$$

giving

$$\begin{bmatrix} V_0^*\Pi_v^{-1}V_0 & -I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{m}(0) \\ \hat{\lambda}(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ V_L^*\Pi_v^{-1}V_L & I \end{bmatrix}\begin{bmatrix} \hat{m}(L) \\ \hat{\lambda}(L) \end{bmatrix} = 0$$

If we choose

$$\theta_f(0) = V_0^* \Pi_v^{-1} V_0 \,,\; \theta_b(L) = V_L^* \Pi_v^{-1} V_L$$

then the change of variables using $T(x)$ produces the boundary conditions

$$\Psi_f(0) = \Psi_b(L) = 0$$

so that $\Psi_f = \Psi_f^0$ and $\Psi_b = \Psi_b^0$. Therefore

$$\hat{m}(x) = P(x)\Psi_f^0(x) + P(x)\Psi_b^0(x)$$

$$\hat{\lambda}(x) = \theta_f(x)P(x)\Psi_b^0(x) - \theta_b(x)P(x)\Psi_f^0(x)$$

Furthermore,by proceeding in the same fashion with Eq. (6.1c), we get

$$e_f(0) = V_0^* \Pi_v^{-1} v \,,\; e_b(L) = V_L^* \Pi_v^{-1} v$$

so that

$$e_f(x) = \Phi_{A_f}(x,0)\, V_0^* \Pi_v^{-1} v + e_f^0(x)$$
$$e_b(x) = \Phi_{A_b}(x,L)\, V_L^* \Pi_v^{-1} v + e_b^0(x)$$

It follows that

$$\Theta_{12}(x) = \Theta_{21}(x) = 0$$

$$\Theta_{11}(x) = \Phi_{A_f}(x,0)\theta_f(0)\Phi_{A_f}^*(x,0) + \Pi_f(x) = \theta_f(x)$$
$$\Theta_{22}(x) = \Phi_{A_b}(x,L)\theta_b(L)\Phi_{A_b}^*(x,L) + \Pi_b(x) = \theta_b(x)$$

Therefore, $P_m(x, j\omega) = P(x)$.

8. Algorithm Complexity

We will examine the computational complexity of the algorithm presented in Section 5. For computational purposes, we assume that the interval $[0,L]$ is partitioned into X regions , and that an FFT with Ω frequencies is used. Table

1 gives the complexities of the major steps necessary to compute the estimate $\hat{m}(\cdot,\cdot)$.

TABLE 1 Complexity of the Smoothing Algorithm

Step	Complexity
Fourier transform the observations	$O(\Omega\ N_s p \log\Omega\ )$
Compute $\theta_f,\theta_b,\Psi_f,\Psi_b$	$O(\Omega\ [Xn^3+N_s np+N_s n^2])$
Compute $\hat{m}(x,j\omega)$	$O(\Omega\ Xn^2)$
Inverse transform $\hat{m}(x,j\omega)$	$O(\Omega\ nX\log\Omega\ )$

As can be seen from Table 1, the overall complexity of the algorithm is

$$O(\Omega\ [X(n^3+n\log\Omega\ )+N_s(np+n^2+p\log\Omega\ )])$$

For comparison, if a Wiener smoother is used

$$\hat{m}(x,t) = \mathcal{F}^{-1}\{S_{mY}(x,j\omega)S_{YY}^{-1}(j\omega)\,Y(j\omega)\}$$

where

$$Y(j\omega) = [y_1^T(j\omega),\ \cdots\ ,y_{N_s}^T(j\omega)]^T$$

$S_{mY}(x,j\omega)$ is the cross-spectral density between $m(x,j\omega)$ and $Y(j\omega)$, and $S_{YY}(j\omega)$ is the spectral density of $Y(j\omega)$; then the complexity of the reconstruction is

$$O(\Omega\ [N_s p(nX+N_s^2p^2+\log\Omega\ )+nX\log\Omega\ ])$$

An example using this type of approach can be found in [11]. We see that the algorithm presented in this paper is most advantageous when the number of sensors are large (linear in N_s versus cubic in N_s).

9. Example: Sound Waves in an Air Filled Pipe

As an example of the use of our algorithm, we consider the problem of estimating the particle velocity and sound pressure levels inside an air filled pipe with rigid walls, given observations of sound pressure levels at discrete points along the pipe. We are interested in estimating the velocities and pressures in the section of the pipe from $x = 0$ to $x = 3$ meters, and we assume that forward traveling waves enter from the $x = 0$ boundary, and backward traveling waves enter from the $x = 3$ boundary. The particle displacement ψ is assumed to satisfy

$$\psi_{tt}(x,t) = c^2\psi_{xx}(x,t)+\epsilon(x,t) \ , \ x \in (0,3)$$

$$\psi_t(0,t) - c\psi_x(0,t) = d_1(t)$$

$$\psi_t(3,t)+c\psi_x(3,t) = d_2(t)$$

where c is 332 meters/second , ϵ is a noise term accounting for yielding of the pipe walls, and d_1 and d_2 are waves entering the pipe section. The observations are

$$y_j(t) = -\rho_0 c^2\psi_x(x_j,t)+w_j(t) \ , \ j = 1, \cdots , N_s$$

where ρ_0 is the density of air (1.29 kilogram/meter ^{3}), and $w_j(t)$ is observation noise. In DH form this system becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} m_1(x,t) \\ m_2(x,t) \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} m_1(x,t) \\ m_2(x,t) \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon(x,t) \end{bmatrix}$$

$$[-1 \vdots 1]\begin{bmatrix} m_1(0,t) \\ m_2(0,t) \end{bmatrix} = d_1(t) \ , \ [1 \vdots 1]\begin{bmatrix} m_1(3,t) \\ m_2(3,t) \end{bmatrix} = d_2(t)$$

$$y_j(t) = [-\rho_0 c \vdots 0] \begin{bmatrix} m_1(x_j,t) \\ m_2(x_j,t) \end{bmatrix} + w_j(t) \quad j = 1, \cdots, N_s$$

where

$$m_1(x,t) = c\psi_x(x,t)$$

$$m_2(x,t) = \psi_t(x,t)$$

We make the following statistical assumptions

$$R_w^{-1}(j\omega) = 10^{-2} \;\; pascals^2$$

$$\Pi_v(j\omega) = 10^{-6} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \; meters^2/seconds^2$$

$$Q(j\omega) = 10^{-2} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \; meters^2/seconds^4$$

for $\omega = 2\pi l$ *radians/second*;$l = \pm 1, \pm 2, \cdots, \pm 500$. Figures 1 and 2 show the smoothing error covariance for the sound pressure and particle velocity respectively, as a function of the number of sensors (N_s) uniformly distributed along the pipe. Figures 3 and 4 show the smoothing error covariances as a function of frequency for a pipe with 5 sensors. In these figures, the x-axis is frequency (from 0 to 1000 π radians /second) and the y-axis is length along the pipe from 0 to 3 meters. One can see the effects of spatially sampling the sound field. In Figure 4, the error covariance maximum occurs at $\omega = 664\pi$ radians /second. Since the spatial sampling frequency for 5 sensors is 4π radians / meter the error covariance maximum occurs when the wavenumber $k = \dfrac{\omega}{c}$ of the sound waves matches the Nyquist sampling frequency. Figure 5 shows the actual and reconstructed time waveforms for the sound pressure level, using 8 sensors. The actual and reconstructed pressure field as a function of space at the frequency $\omega = 500\pi$ radians /second appears in Figure 6. Figure 7 displays the actual and reconstructed particle velocity as a function of space at the same frequency.

10. Concluding Remarks

The input estimates can be interpreted as the result of a generalized Born inversion procedure. For instance, in the case of the 1D wave equation, $\epsilon(x, j\omega)$ will be the Born approximation to the wave speed variations in an inverse scattering experiment [7]. In such problems, one may update the wave speed function in an iterative fashion. The approach used in this paper to derive the smoothing algorithms is based on using a frequency domain two point boundary value problem to describe the system's dynamics. A related approach to characterizing a vibrating system's dynamics is given in [6], where variations in the system's parameters are assumed to occur at discrete points along it's length, giving rise to a constant diagonal A matrix . This type of model can be handled using the algorithms developed in this paper. In both of these approaches, one can interpret the boundary conditions at the endpoints of the system as describing the reflection and transmission coefficients of the hyperbolic system. In the DH case, the reflection and transmission coefficients arise in a natural way when the A matrix is diagonal. Note that for many dissipative systems, there does not exist a discrete set of spatial eigenfunctions, so that a modal expansion of the dynamics and observations, a technique used quite often in distributed parameter filtering and control, is not in general applicable to the systems discussed in this paper. The frequency domain description of hyperbolic systems with 2 and 3 spatial dimensions involves distributed parameter acausal linear systems. Efficient smoothing algorithms for the 2-D wave equation, for example, can be developed in this way [10].

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Appendix A

To study the case when $\Lambda(x)$ is not invertible, we put Eqs (2.1)-(2.3) into a canonical form which separates the distributed parameter states from the so-called local states. The local state variables typically correspond to damping forces acting on the distributed parameter system. These damping forces may be due to external inputs, corresponding to an active control system, or the forces may be passive, such as structural damping in a beam. Phillips [9] proves the existence of a family of orthogonal matrices $\{U(x)\ ;\ 0 \leq x \leq L\}$ with absolutely continuous elements having square integrable derivatives such that

$$U'(x)\Lambda(x)U(x) = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(x) \end{bmatrix}$$

where $\Lambda_{22}(x)$ is positive definite and r by r. Phillips gives an explicit algorithm for calculating $U(x)$. With the change of variables

$$y(x,t) = U'(x)m(x,t)$$

Eqs. (2.1)-(2.3) become

$$\frac{\partial}{\partial t}y(x,t) = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(x) \end{bmatrix}\frac{\partial}{\partial x}y(x,t)+\{U'\Lambda U_x+U'GU\}y(x,t)+U'(x)\epsilon(x,t)$$

$$H_0U(0)y(0,t) = d_1(t) \ , \ H_L U(L)y(L,t) = d_2(t)$$

$$y(x,t_0) = U'(x)m_0(x)$$

The stability assumptions are unchanged, because

$$U'\Lambda U_x+U'_x\Lambda U+U'G'U+U'GU-\frac{\partial}{\partial x}\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(x) \end{bmatrix} = U'(G+G'-\frac{\partial}{\partial x}\Lambda)U \leq 0$$

$$y'(L,t)\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(L) \end{bmatrix}y(L,t) = m'(L,t)\Lambda(L)m(L,t)$$

$$y'(0,t)\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(0) \end{bmatrix}y(0,t) = m'(0,t)\Lambda(0)m(0,t)$$

We therefore assume that Eqs (2.1)-(2.3) have the following canonical form

$$\frac{\partial}{\partial t}\begin{bmatrix} m_1(x,t) \\ m_2(x,t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(x) \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} m_1(x,t) \\ m_2(x,t) \end{bmatrix}+\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} m_1(x,t) \\ m_2(x,t) \end{bmatrix} + \begin{bmatrix} \epsilon_1(x,t) \\ \epsilon_2(x,t) \end{bmatrix} \quad \text{(A.1)}$$

$$\begin{bmatrix} H_{01} \vdots H_{02} \end{bmatrix}\begin{bmatrix} m_1(0,t) \\ m_2(0,t) \end{bmatrix} = d_1(t) \ , \ \begin{bmatrix} H_{L1} \vdots H_{L2} \end{bmatrix}\begin{bmatrix} m_1(L,t) \\ m_2(L,t) \end{bmatrix} = d_2(t) \quad \text{(A.2)}$$

$$m_0(x) = 0$$

Fourier expanding Eqs (A.1),(A.2) at $\omega = 2\pi l/T$, $l = 0, \pm 1, \cdots$, we get

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(x) \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} m_1(x,j\omega) \\ m_2(x,j\omega) \end{bmatrix} = \begin{bmatrix} j\omega I - G_{11} & -G_{12} \\ -G_{21} & j\omega I - G_{22} \end{bmatrix} \begin{bmatrix} m_1(x,j\omega) \\ m_2(x,j\omega) \end{bmatrix} - \begin{bmatrix} \epsilon_1(x,j\omega) \\ \epsilon_2(x,j\omega) \end{bmatrix} \tag{A.3a}$$

$$\begin{bmatrix} H_{01}(j\omega) & \vdots & H_{02}(j\omega) \end{bmatrix} \begin{bmatrix} m_1(0,j\omega) \\ m_2(0,j\omega) \end{bmatrix} = d_1(j\omega) \tag{A.3b}$$

$$\begin{bmatrix} H_{L1}(j\omega) & \vdots & H_{L2}(j\omega) \end{bmatrix} \begin{bmatrix} m_1(L,j\omega) \\ m_2(L,j\omega) \end{bmatrix} = d_2(j\omega) \tag{A.3c}$$

The stability assumption (2.4) implies that

$$G_{11} + G_{11}' \leq 0 \tag{A.4}$$

The first row of Eq (A.3a) gives

$$(j\omega I - G_{11}) m_1(x,j\omega) = G_{12} m_2(x,j\omega) + \epsilon_1(x,j\omega) \tag{A.5}$$

From Eq (A.4) we see that the eigenvalues of G_{11} have negative or zero real parts. The stability assumptions that were discussed in Section 2 eliminate the possibility that these eigenvalues are on the imaginary axis. Solving Eq. (A.5) for $m_1(x,j\omega)$ and substituting into Eqs. (A.3) gives

$$\frac{\partial}{\partial x} m_2(x,j\omega) = \Lambda_{22}^{-1} [j\omega I - G_{22} - G_{21}(j\omega I - G_{11})^{-1} G_{12}] m_2(x,j\omega) - \Lambda_{22}^{-1} (G_{21}(j\omega I - G_{11})^{-1} \epsilon_1 + \epsilon_2) \tag{A.6}$$

$$[H_{01}(j\omega I - G_{11}(0))^{-1} G_{12}(0) + H_{02}] m_2(0,j\omega) = d_1(j\omega) \tag{A.7a}$$

$$[H_{0L}(j\omega I - G_{11}(L))^{-1}G_{12}(L)+H_{L2}]m_2(L,j\omega) = d_2(j\omega) \quad \text{(A.7b)}$$

In deriving Eqs. (A.7) we have assumed that $\epsilon(0,j\omega) = \epsilon(L,j\omega) = 0$. These equations are now in the form of Eqs (3.2),(3.4).

Appendix B

In this appendix we will show that $y_c(\cdot)$ and θ defined in Eqs (4.3) span $Y^{\perp}$, the subspace of random variables orthogonal to Y, so that $Y \oplus Y^{\perp}$ is a direct sum decomposition of H, the underlying Hilbert space generated by $\{\mu\ ,\ \rho(x)\ ,\ 0 \le x \le L\ ,\ w_j\ ,\ j=1,\cdots N_s\}$. We introduce η_j and z_j defined by

$$\eta_j = R_w^{-1/2}w_j\ ,\ z_j = R_w^{-1/2}y_j$$

so Eq. (4.2) can be rewritten in an obvious operator notation as

$$z = \mathbf{F}\mu + \mathbf{G}\rho + \eta \quad \text{(B.1)}$$

where

$$z = [z_1'z_2' \cdots z_{N_s}']' \quad \eta = [\eta_1'\eta_2' \cdots \eta_{N_s}']'$$

If $a \in H$, its projection onto Y is denoted $\hat{a}$ and its projection onto $Y^{\perp}$ is $\tilde{a}$. Decomposing $(\mu\ ,\ \rho\ ,\ \eta)$ gives

$$\mu = \hat{\mu}+\tilde{\mu}\ ,\ \rho = \hat{\rho}+\tilde{\rho}\ ,\ \eta = \hat{\eta}+\tilde{\eta}$$

where

$$\hat{\eta} = R_z^{-1}z\ ,\ \hat{\rho} = \mathbf{G}^*\hat{\eta}\ ,\ \hat{\mu} = \mathbf{F}^*\hat{\eta} \quad \text{(B.2)}$$
$$R_z = [\mathbf{F}\mathbf{F}^* + \mathbf{G}\mathbf{G}^* + I]$$

Therefore,

$$\tilde{\rho} = \rho - \mathbf{G}^*\eta + \mathbf{G}^*\tilde{\eta}\ ,\ \tilde{\mu} = \mu - \mathbf{F}^*\eta + \mathbf{F}^*\tilde{\eta}$$

If we define

$$y_c = \rho - \mathbf{G}^*\eta\ ,\ \theta = \mu - \mathbf{F}^*\eta$$

then

$$y_c = \tilde{\rho} - \mathbf{G}^* \tilde{\eta} = (I + \mathbf{G}^* \mathbf{G}) \tilde{\rho} + \mathbf{G}^* \mathbf{F} \tilde{\mu} \tag{B.3a}$$

$$\theta = \tilde{\mu} - \mathbf{F}^* \tilde{\eta} = \mathbf{F}^* \mathbf{G} \tilde{\rho} + (I + \mathbf{F}^* \mathbf{F}) \tilde{\mu} \tag{B.3b}$$

where we have used the fact (see Eq. (B.1))

$$0 = \mathbf{F}\tilde{\mu} + \mathbf{G}\tilde{\rho} + \tilde{\eta} \tag{B.4}$$

It is clear from Eqs. (B.3) - (B.4) that $(y_c \ , \ \theta)$ uniquely determine $(\tilde{\mu} \ , \ \tilde{\rho} \ , \ \tilde{\eta})$ thus $(y \ , \ y_c \ , \ \theta)$ uniquely determine $(\mu \ , \ \rho \ , \ \eta)$. Note also that Eq. (B.3) implies $y_c \in Y^{\perp}$ and $\theta \in Y^{\perp}$. To verify Eqs. (4.3)-(4.4) one need only evaluate $\mathbf{F}^*$ and $\mathbf{G}^*$.

Figure 1 - Smoothing error for pressure

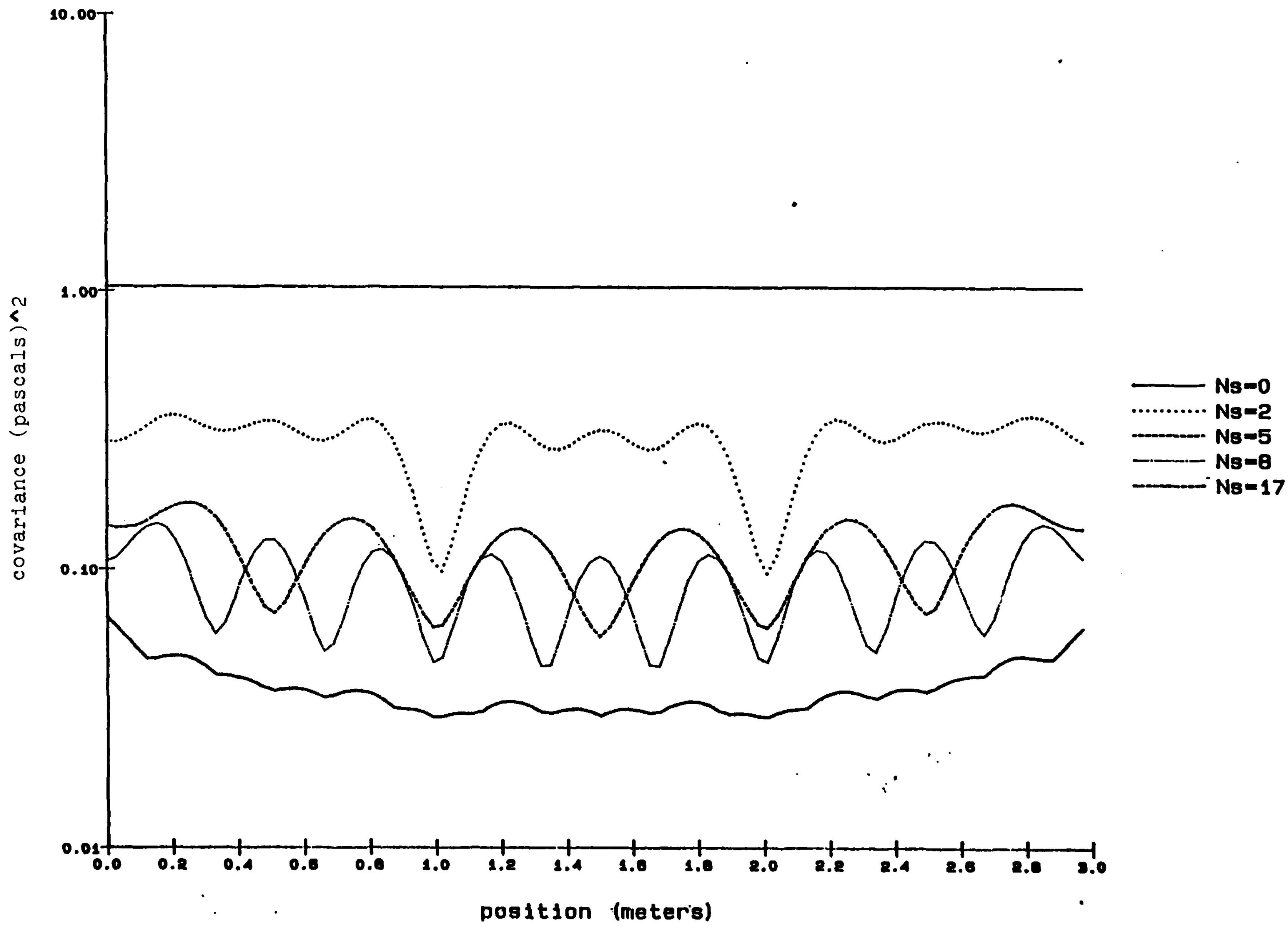


Figure 2 - Smoothing error for velocity

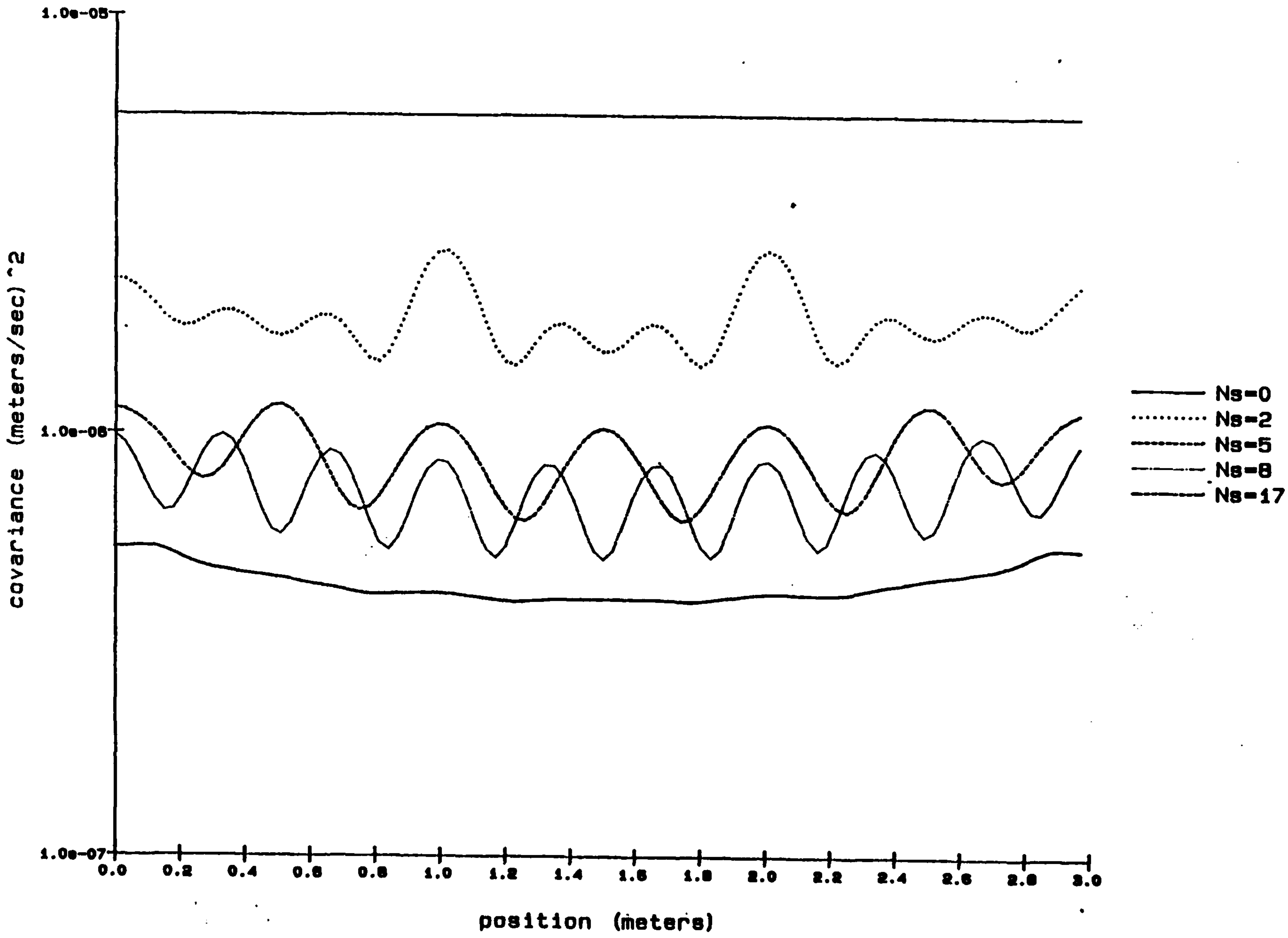


Figure 3 - Smoothing error for pressure as a function of frequency

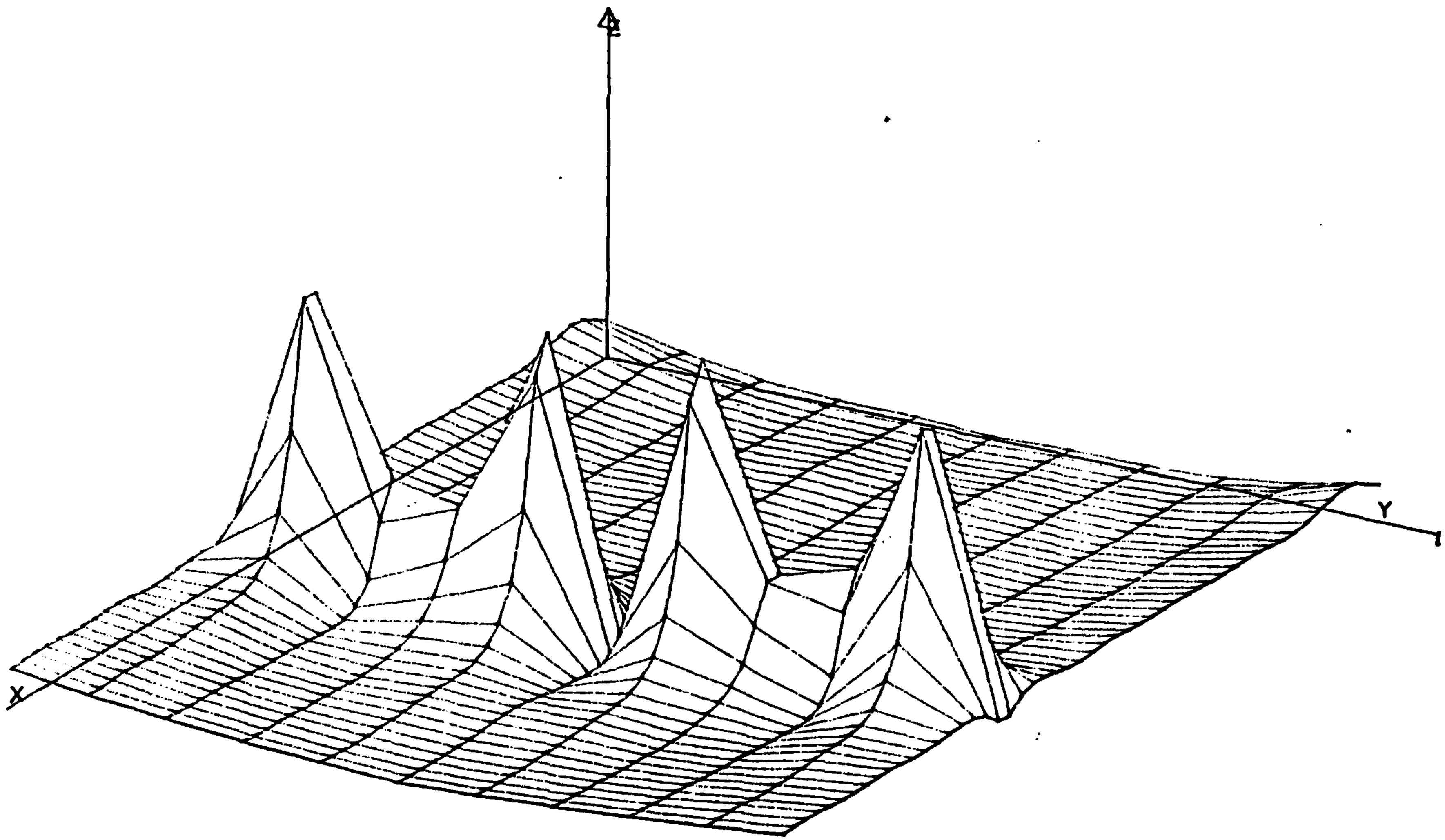


Figure 4 - Smoothing error for velocity as a function of frequency

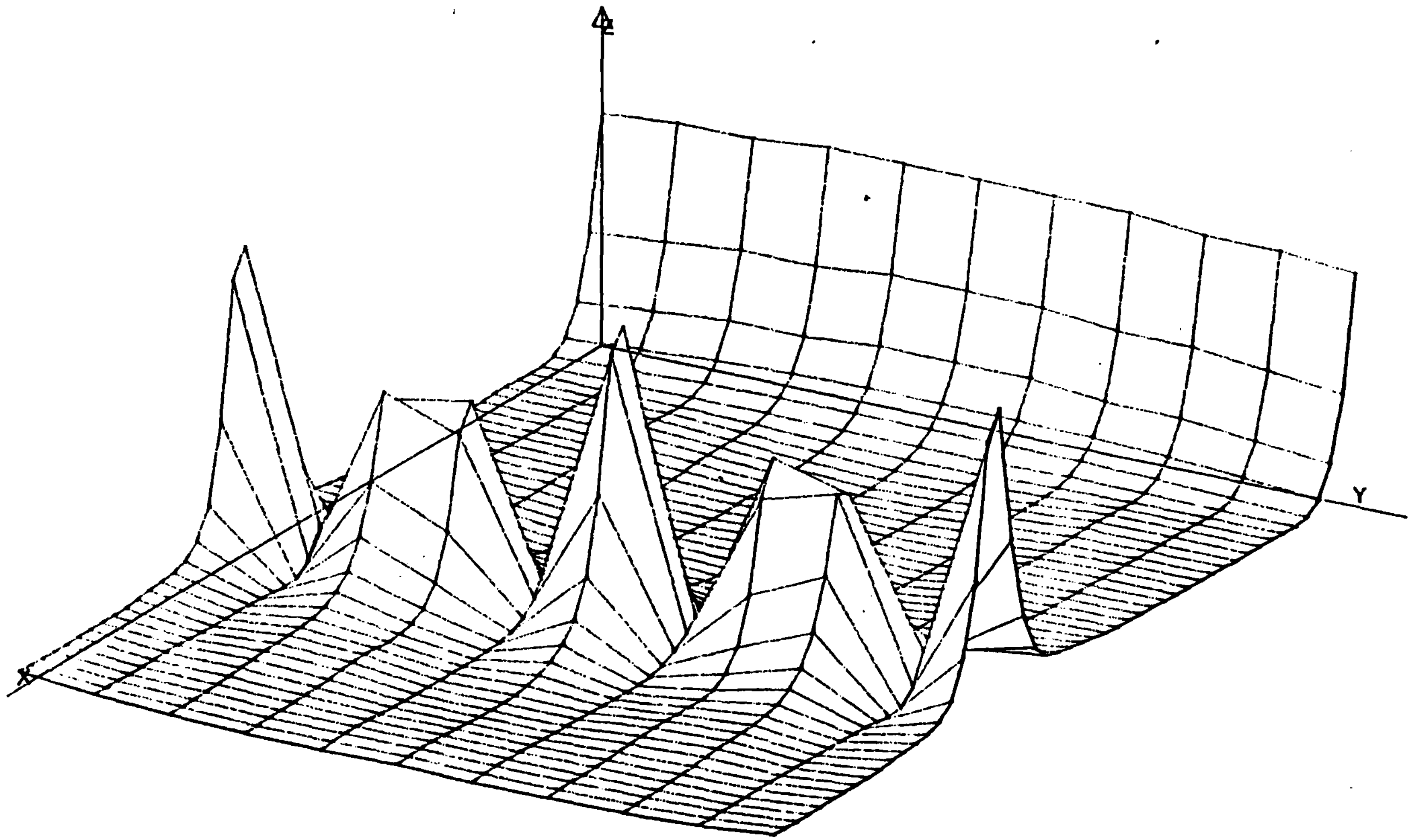


Figure 5 - Pressure at x = 1.5 meters

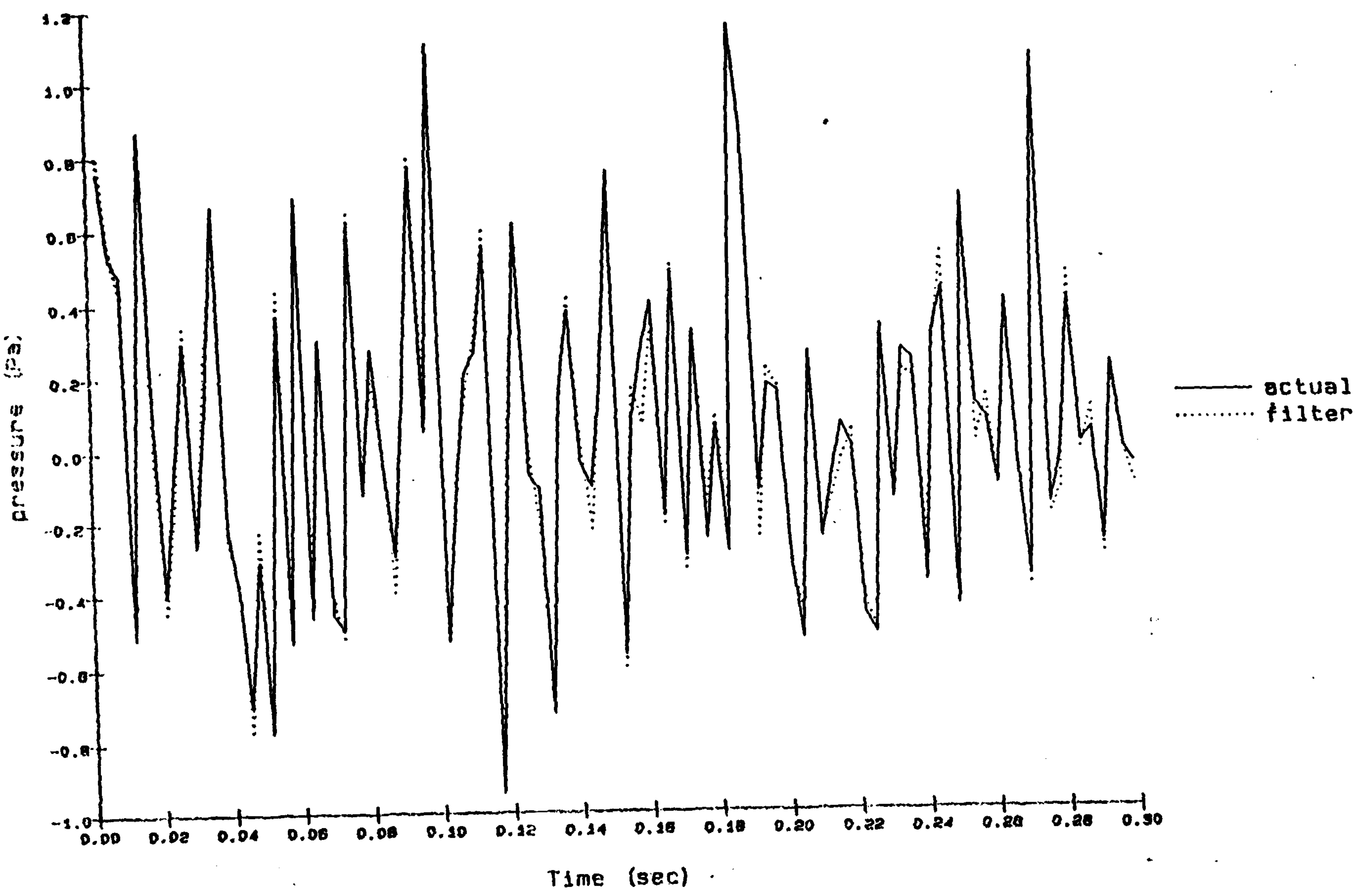


Figure 6 - Pressure at 1570 rads/sec

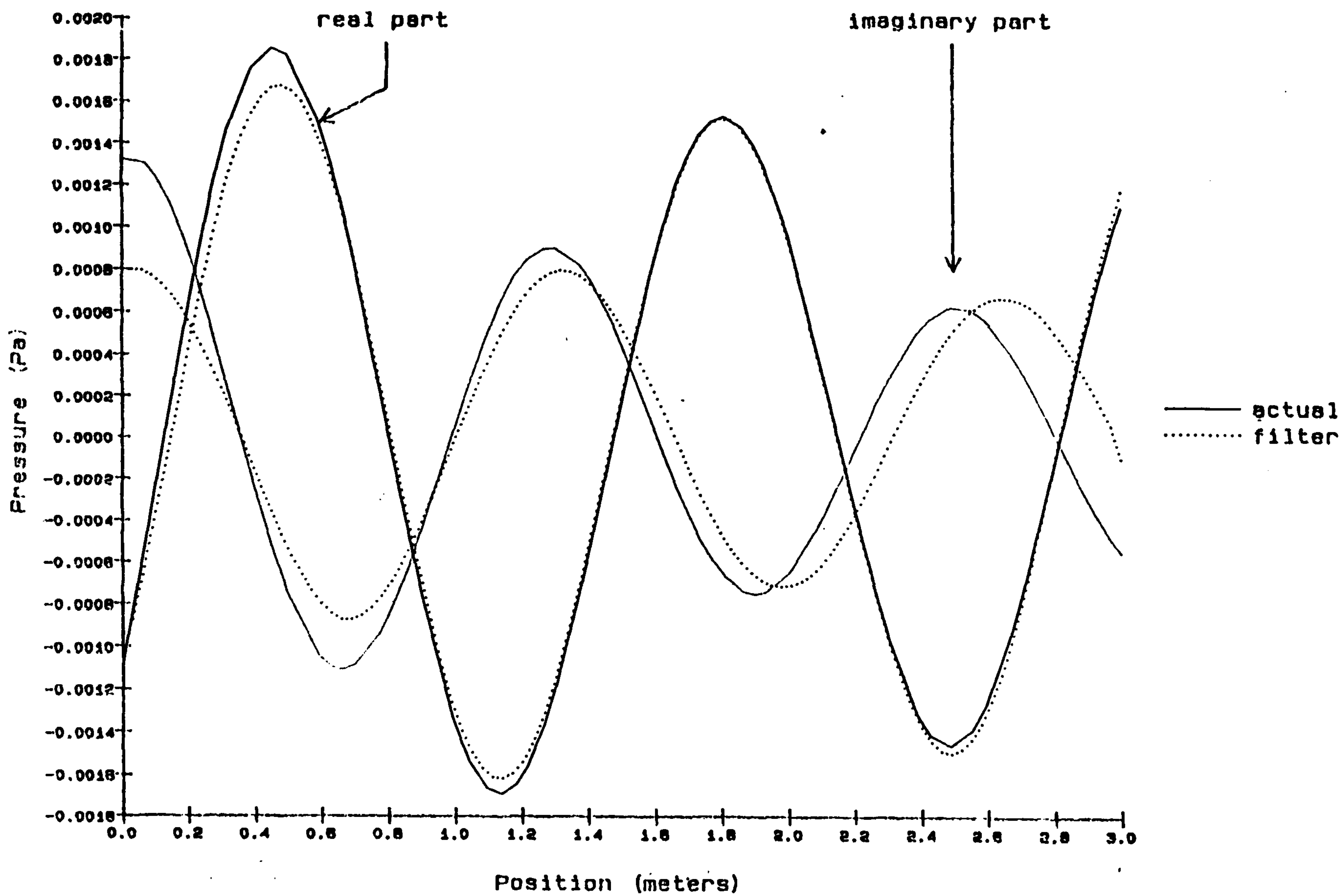
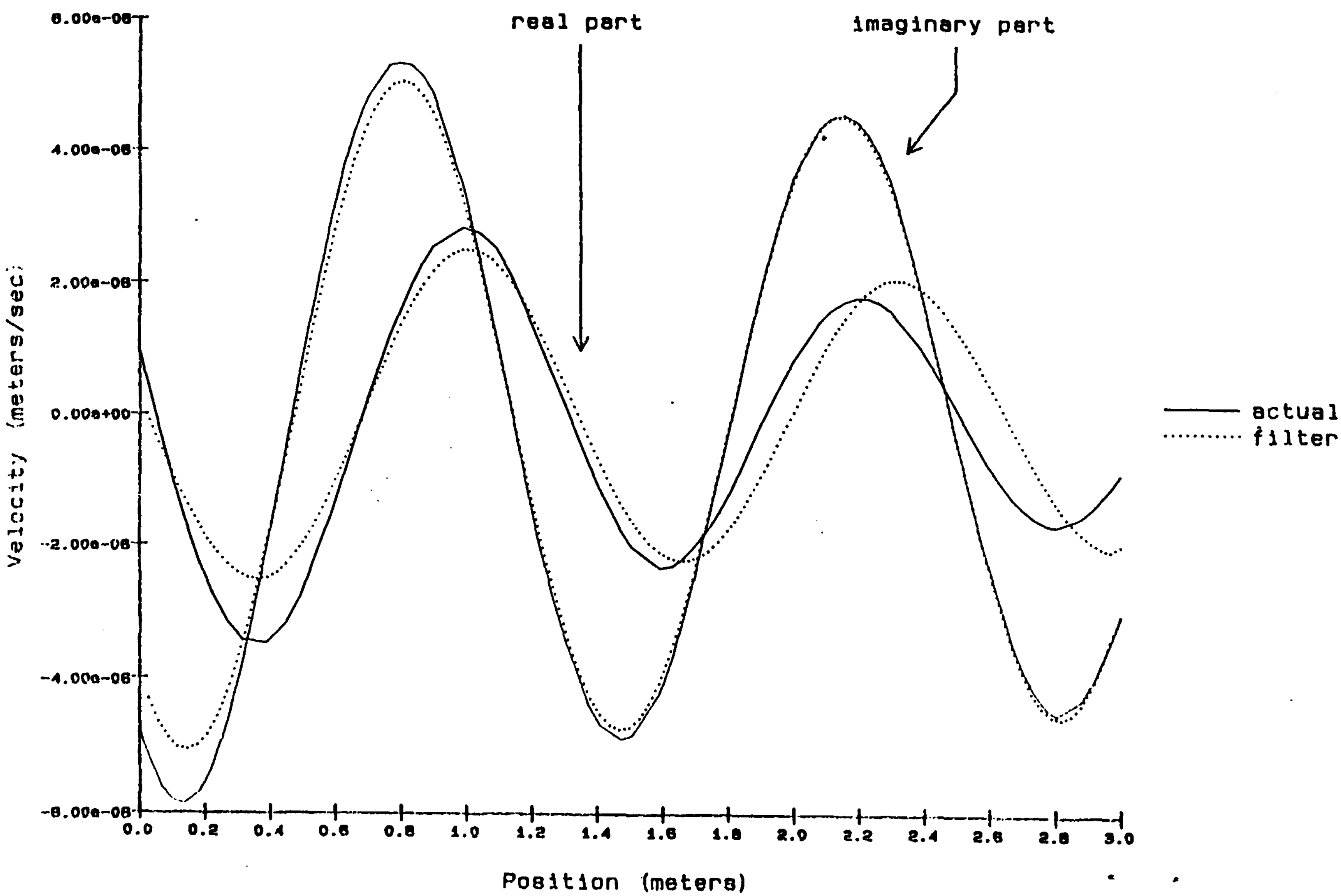


Figure 7- Velocity at 1570 rads/sec



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This paper presents an efficient method of smoothing steady-state, dissipative hyperbolic systems with one spatial dimension. The observations are from point sensors placed on the system. ~~We show that~~ under realistic stability conditions there exists a family of finite-dimensional acausal linear systems that characterize the frequency domain behavior of the hyperbolic system. Using this characterization, one ~~we~~ develops a smoothing algorithm that is recursive with respect to the sensors, resulting in a significant decrease in computational complexity relative to other methods. ~~We illustrate~~ the algorithm's performance by studying the smoothing problem for sound waves in an air-filled pipe.

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